Moduli Spaces of Tropical Curves

ICERM Bootcamp

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Recall: \( \mathbf{M}_g = \text{algebraic variety} \) (scheme, stack) 
\[ (g \geq 2) \]

\[ \mathbf{M}_g(k) = \{ \text{smooth projective } k\text{-curves of genus } g \}/\Xi \]

\[ \mathbf{M}_g(\mathbb{C}) = \mathcal{T}_g / \text{Mod}(S_g) \]

Teichmüller Space

\( = \text{hyperbolic manifolds } - S_g \)

\[ H^*(\mathbf{M}_g(\mathbb{C})) \]

Mapping Class Group
From this morning (M. Brandt)

\[ k = \text{algebraically closed valued field}. \]

\[ R \subset k \text{ valuation ring, } k = \text{residue field}. \]

Each \( X \in \text{Mg}(k) \) has a unique stable model

\[ X_k = X \quad \text{and} \quad \omega_X = (\text{rel.}), \text{ample}. \]

\[ \triangleright \text{abstract tropical curve} \]

\( G = \text{dual graph of } X_k \) \{ 
\[ V(G) = \text{irreducible components} \]
\[ E(G) = \text{nodes} \]
\[ l : E(G) \to \mathbb{R}_{>0} \] "Thickness of node"

"Thickness of \((xy - t = 0) := \text{val}(t)\)"
Note:
- Ampleness of $\omega_x$ depends only on $G$.
- $\omega_x$ ample $\iff$ every vertex of genus $0$ has valence at least $3$.
- Only finitely many such stable graphs in each $g$. 

\[ g = 2 \]

![Diagram of graphs labeled by genus]
Def: \( M_{\text{g}}^{\text{top}} = \{ \text{stable tropical curves of genus } g \} / \cong \)

\[
= \bigcup_{G} \frac{E(G)}{\text{Aut}(G)}
\]

Note: \( M_{\text{g}}^{\text{top}} \) is a cone over \( \Delta_{\text{g}} := \{ (G,l): \sum_{e} l(e) = 1 \} \).

Example: \( g = 2 \)

\[
\begin{align*}
\text{Diagram:} \\
&\text{Diagram:} \\
&\text{Diagram:}
\end{align*}
\]
Marked Points: Fix $g, n \geq 0$ such that $2g - 2 + n > 0$.

$$M_{g,n}(K) = \frac{\{ \text{sm. proj. K-curves of genus } g \text{ w/ } n \text{ distinct marked pts} \}}{\Delta_2}$$
Note: $2g-2+n > 0 \iff \omega_X(p_1+\ldots+p_n) \text{ ample.}$

**Stable reduction:**

$(X, p_1, \ldots, p_n)$ has a unique stable model

$(X, \sigma_1, \ldots, \sigma_n)$

- $\mathcal{X}_k = X$ and $\sigma_i = \overline{p_i}$
- $\omega_{\mathcal{X}}(\sigma_1+\ldots+\sigma_n) = (\text{rel.}) \text{ample}$

Dual graph of $\mathcal{X}_k$: stable $n$-marked tropical curve

**Ex:** $g=1, n=2$
**Stable**: each vertex of genus $0$ has valence $\geq 3$

**Note**: finitely many dual graphs for each $g,n$.

\[ M_{g,n}^{\text{tryp}} = \{ \text{stable } n\text{-marked tropical curves of genus } g \} \sim \]

= cone over $\Delta_{g,n} = \{ (c,l) : \sum_{e} l(e) = 1 \}$

**Example**: $g=1, n=2$

Diagram:

- Blue and orange graphs depict different dual graphs for $g=1, n=2$.
A map \( \lambda : M_g(C) \to M_g^{hyp} \) (recall: \( M_g(C) = T_g/\text{Mod}(S_g) \))

\( X \in T_g \) hyperbolic metric

Fix small \( \varepsilon > 0 \).
Let $G$ be the dual graph of nodal curve obtained by contracting all closed geodesics of length $< 3$. 

For $e \in E(G)$, let $\lambda(e)$ be the geodesic of length $ae < 3$.

$\lambda(e) := -\log (ae/3)$.

$$\lambda : X \rightarrow (G, \lambda)$$

**Observation**: $\lambda$ is proper and hence induces

$$\lambda^* : H^*_c(M_{g_p}^{top}) \rightarrow H^*_c(M_{j(C)})$$

**Thm** (via Deligne's MHS): $\lambda^*_Q$ is injective
Equivaleently (via Poincaré duality)

\[ H^{6g-6-k}(\mathcal{M}_g; \mathbb{Q}) \rightarrow \hat{H}_{k-1}(\Delta_g; \mathbb{Q}) \]